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Saffman–Taylor instability in yield-stress fluids

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When a fluid is pushed by a less viscous one the well-known Saffman–Taylor instability phenomenon arises, which takes the form of fingering. Since this phenomenon is important in a wide variety of applications involving strongly non-Newtonian fluids – in other words, fluids that exhibit yield stress – we undertake a full theoretical examination of Saffmann–Taylor instability in this type of fluid, in both longitudinal and radial flows in Hele-Shaw cells. In particular, we establish the detailed form of Darcy's law for yield-stress fluids. Basically the dispersion equation for both flows is similar to equations obtained for ordinary viscous fluids but the viscous terms in the dimensionless numbers conditioning the instability contain the yield stress. As a consequence the wavelength of maximum growth can be extremely small even at vanishing velocities. Additionally an approximate analysis shows that the fingers which are left behind at the beginning of destabilization should tend to stop completely. Fingering of yield-stress fluids therefore has some peculiar characteristics which nevertheless are not sufficient to explain the fractal pattern observed with colloidal systems.

1. Introduction

In recent years it has become apparent that some peculiar phenomena can occur when a viscous fluid is displaced by a more viscous one under conditions for which the usual Saffman–Taylor (1958) instability (see the review by Homsy 1987) was expected. In particular, experiments with radial Hele-Shaw cells using non-Newtonian fluids have shown striking qualitative differences in the fingering pattern (see for example the review by Van Damme *et al.* 1994 or McCloud & Maher 1995). It was discovered that, when the high-viscosity fluid is viscoelastic, the interface grows along a narrow and very tortuous finger leading to branched, fractal patterns (Nittman, Daccord & Stanley 1985). It was also shown that this viscous fingering pattern can be replaced by a viscoelastic fracture pattern for appropriate Deborah numbers (Lemaire *et al.* 1991). Since these patterns occur either for immiscible or miscible fluids the role of surface tension remains unclear (Van Damme 1989). On the other hand the viscous properties of the fluids seem to be decisive. In general colloidal fluids were used, which are known to be viscoelastic and shear-thinning. As a consequence the Saffman–Taylor instability problem was revisited for such fluids. Wilson (1990) considered an Oldroyd-B fluid which exhibits elasticity and the case of power-law fluids was treated by Wilson (1990) for unidirectional flows and by Sader, Chan & Hughes (1994) and Kondic, Palffy-Muhoray & Shelley (1996) for radial flows. However, except in the case of fluids with a negative viscosity for which slip layers may form (Kondic *et al.* 1996), the corresponding theoretical results did not show strong changes in the basic process of instability compared to Newtonian fluids. For viscoelastic fluids, Wilson (1990) found a kind of resonance which can produce sharply increasing (in fact unbounded) growth rates as the relaxation time of the fluid increases. However Sader *et al.* (1994) argued that Wilson considered the regime of large Deborah numbers, where elastic effects are important, which does not correspond to practical situations. Sader *et al.* (1994) mainly showed that decreasing the power-law index dramatically increases the growth rates of perturbation at the interface and provides effective length compression for the formation of viscous-fingering patterns, thus enabling them to develop much more rapidly.

From a rheological point of view the main characteristics of many concentrated systems are thixotropy and yield stress. In the present work we shall set aside the problem of thixotropy, assuming that the characteristic time of viscosity change of the material is much smaller than the characteristic flow time. Most natural and industrial materials (glues, inks, pastes, slurries, paints, muds, fresh concrete, etc.) obtained by suspending a large number of particles interacting via colloidal forces or direct contact in water are non-Newtonian fluids exhibiting a yield stress (τ_c), which needs to be overcome for flow to take place (Bird, Dai & Yarusso 1982; Coussot 1997). This yield stress is in fact the strength necessary to break the continuous network of interactions between particles throughout the sample. Thixotropy may be associated to the time required for the structure to restore or break. A typical characteristic of yield-stress fluids is that they give rise to thick deposits (stationary volumes) on steep slopes whereas unyielding fluids go on flowing under gravity as long as surface tension effects remain negligible. Both Van Damme (1989) and Wilson (1990) suggested that taking into account the yield stress of these fluids could be decisive. Here we propose a complete treatment of the Saffman–Taylor instability for yield-stress fluids under the conventional lubrication approximation, both for unidirectional and radial flows.

The instability of radial flows of Newtonian fluids in Hele-Shaw cells has been studied by Bataille (1968), Wilson (1975), and Paterson (1981) who used the vectorial form of Darcy's law. Our treatment is rather similar to the one adopted by Wilson (1990) or Sader *et al.* (1994) who considered power-law fluids and could not directly use such a form of Darcy's law (though Kondic *et al.* 1996 later proposed an approach of this type): (i) within the framework of the lubrication approximation the velocity component perpendicular to the cell plane is neglected even close to the front; (ii) we assume that the uniform velocity distribution is slightly perturbed as a result of front perturbation, from which the stress components are deduced; (iii) the boundary conditions integrated over the fluid depth then give the condition for instability. We shall see that the second assumption is dubious for a yield-stress fluid but should finally lead to acceptable results.

In § 2 we consider the constitutive equation to be taken into account when dealing with complex flows of yield-stress fluids. Thixotropy and elastic effects are neglected. In § 3 we first establish the velocity distribution for a stable flow, which, after integration over the fluid depth and inversion, makes it possible to derive an expression of Darcy's law for yield-stress fluids. Then we consider the unstable flow of a yield-stress fluid pushed by another in one specific direction (§ 4) and radially (§ 5).

2. Constitutive equation of yield-stress fluids

The real, physical existence of yield stress has been the subject of numerous discussions (see for example Barnes & Walters 1985; Astarita 1990; De Kee & Chan Man Fong 1993; Spaans & Williams 1995). In particular the discussion concerned the question of whether so-called viscoplastic fluids have a real yield stress below which they can be considered as solids or simply exhibit a very high viscosity at extremely low shear stress levels. At the very least, it was concluded that yield stress is a practical, engineering reality. We shall simply consider that, for such a fluid, under common conditions of observation, there generally exists an abrupt change in behaviour around a given shear stress value, that we can call the yield stress. Below this critical stress value the fluid is deformed in an essentially elastic manner. Above this critical value the fluid flows. There is a fundamental difference between this type of model and a so-called bi-viscous model often used in numerical approaches in order to avoid the problem of determining the unsheared regions. Indeed, with the bi-viscous model, flow can be observed in the laboratory even for low shear stress, unlike experimental observations (Coussot, Leonov & Piau 1993). Although the bi-viscous model can give realistic results under some flow conditions, it is not relevant when one considers flow situations for which the unsheared parts can play a major role in the observed phenomena.

The elastic behaviour before yielding may be taken into account in the constitutive equation of yield-stress fluids (Doraiswamy *et al.* 1991; Coussot *et al.* 1993). However, since the critical strain before flow is in general small, elastic properties may reasonably be neglected in a first approximation. In addition we shall neglect elastic effects during flow. The Bingham model has long been used to represent experimental data within one or two decades of the shear rate. However, it has recently been shown that the constitutive equation in simple shear of various yielding suspensions can be represented by a Herschel–Bulkley model within a relatively wide shear rate range (Nguyen & Boger 1983; Atapattu, Chhabra & Uhlherr 1995; Doraiswamy *et al.* 1991; Coussot & Piau 1994; Sherwood 1994; Coussot 1995). A three-dimensional expression of the constitutive equation of a (incompressible) Herschel–Bulkley fluid is (Schowalter 1978; Chen & Ling 1996; Coussot 1997):

$$\boldsymbol{D} = 0 \text{ when } -T_{II}^{1/2} < \tau_c \text{ and the stress tensor is indeterminate,} \tag{1a}$$

$$\boldsymbol{T} = \left(\frac{\tau_c}{-D_{II}^{1/2}} + \frac{2^n K}{(-D_{II}^{1/2})^{1-n}} \right) \boldsymbol{D} \text{ otherwise,} \tag{1b}$$

with $D_{II} = -\mathrm{tr}\,\boldsymbol{D}^2/2$ and $T_{II} = -\mathrm{tr}\,\boldsymbol{T}^2/2$, where $\boldsymbol{D}$ is the strain rate tensor, $\boldsymbol{T}$ the extra stress tensor, and K and n two fluid parameters. The Bingham model is obtained by taking $n = 1$. In the following we shall assume that the behaviour of yield-stress fluids can be correctly represented by (1).

3. Stable flow of a yield-stress fluid between two parallel plates

Let us consider the flow of an incompressible yield-stress fluid between two parallel plates. We use a frame of reference (x, y, z) where x is the direction of the flow and y is perpendicular to the planes (Figure 1). The distance separating these planes is $2b$. We assume that inertia is negligible. We shall describe the motion within the framework of the long-wave (lubrication) approximation: the fluid thickness ($2b$) is much smaller than the flow width along z, D, and the length of the fluid in direction x, L; the velocity components parallel to the plane (u and w, respectively along directions x and z) are correspondingly much larger than the velocity component along direction y; the rate of variation of each velocity component is much greater in the direction y than in the perpendicular directions. In particular this assumption means that we neglect the effects of fluid motion along y close to the interface between fluid and air.

For a stable flow, the mobile front (free surface) is a cylindrical surface whose

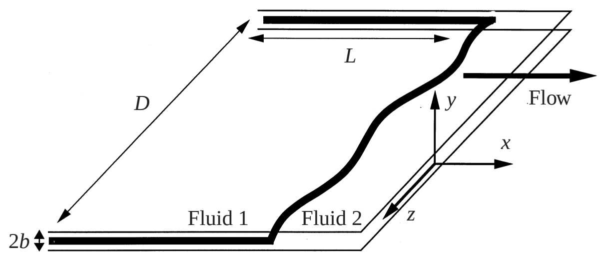

FIGURE 1. Unidirectional flow between two parallel plates: flow geometry and frame of reference.

generatrix remains parallel to Oz; thus the only non-zero velocity component is the one along x, u, which only depends on y. As a consequence it results from (1b) that the only non-zero component of the stress tensor is τ_{xy} which only depends on y. From the equations of motion it results that $p^*(= p + \rho g Z$, where p is the pressure, ρ the fluid density, g the acceleration due to gravity and Z the current level above a fixed horizontal plane) only depends on x and t so that

$$p^* = p_0^*(x,t) = Ax + B, \tag{2}$$

Where A and B only depend on t. In addition the flow is symmetrical with respect to the mid-plane $y = 0$ and considering (1a) it may be shown that

$$u_0'(y) = 0 \quad \text{for} \quad |y| \leqslant y_0 \quad \text{with} \quad y_0 = \frac{\tau_c}{\alpha \partial p^* / \partial x}, \tag{3}$$

where α is equal to -1 for a flow in the direction of positive x values and equal to 1 otherwise. From (1)–(3) and the motion equation along x, assuming no slip at the wall, we deduce the velocity profile:

$$u_0(y) = \frac{\alpha}{m+1}\left(\frac{\alpha}{K}\frac{\partial p^*}{\partial x}\right)^m \left[(|y| - y_0)^{m+1} - (b - y_0)^{m+1}\right] \quad \text{when} \quad y_0 < |y| < b, \tag{4a}$$

$$u_0(y) = -\frac{\alpha}{m+1}\left(\frac{\alpha}{K}\frac{\partial p^*}{\partial x}\right)^m (b - y_0)^{m+1} \quad \text{when} \quad |y| < y_0, \tag{4b}$$

with $m = 1/n$. Since within the framework of the lubrication approximation we neglected the velocity component along direction y and, in particular, close to the front, here we obtain a velocity distribution similar to that of a uniform flow between two parallel plates. In particular this predicts that the fluid remains rigid within a certain region ($|y| \leqslant y_0$). It is well known (Lipscomb & Denn 1984; Piau 1996) that this result cannot be exact for varying flows. The velocity distribution as given by (4) can nevertheless be seen as a good approximation of the real velocity distribution not too close to the front. The above approach will give results closer to reality when, as is precisely the case here, only the relationship between the pressure drop and the mean velocity is sought.

From (4) we can find the mean velocity through a cross-section:

$$U = \frac{\alpha}{m+1}\left(\frac{\alpha}{K}\frac{\partial p^*}{\partial x}\right)^m (b - y_0)^{m+1}\left[\frac{1}{m+2}\left(1 - \frac{y_0}{b}\right) - 1\right]. \tag{5}$$

The boundary conditions, in particular those concerning the pressure, must be expressed in terms of integrals over the fluid depth since (4) is not intended to describe the exact flow characteristics at the boundaries. Within this framework, the complete solution of the problem may be found from the set of equations (2)–(3) and (5) in addition to the boundary conditions in terms of mean velocity or pressure drop.

Equation (5) may be expressed in the following dimensionless form:

$$H_b^{-m} = \frac{G^{-2}(G-1)^{m+1}(mG+G+1)}{(m+1)(m+2)} \tag{6}$$

with $H_b = \tau_c b^n/(K|U|^n)$ and $G = b/y_0$. H_b reflects the ratio of the flow-independent part of viscous dissipations (related to yield stress) to the shear-dependent part. From (3) it results that G expresses the ratio of the potential energy (pressure drop) of a fluid portion to viscous dissipations related to the yield-stress term in the Herschel–Bulkley model. It follows from (6) that

$$G \approx (m+2)^n H_b^{-1} \quad \text{when} \quad G \to \infty, \tag{7}$$

$$G \approx 1 + (m+1)^{1/(m+1)} H_b^{-m/1+m} \quad \text{when} \quad G \to 1. \tag{8}$$

As a consequence, for $G-1$ within a range of a few decades, it is in general possible to find appropriate values for the positive coefficients c and d so that the expression

$$G = 1 + cH_b^{-d} \tag{9}$$

is an approximation of (6) within a few percent. For example, this has been done for uniform free-surface flows of mud suspensions over a wide inclined plane (Coussot 1997). Note that the velocity profile for such a flow is once again given by (4) replacing b by h, the fluid depth, and $\partial p^*/\partial x$ by $-\rho g \sin i$ where i is the channel slope. In the case considered by Coussot (1997) n was equal to 1/3 and expression (9) with $c = 1.93$ and $d = 0.9$ provided an approximation of G within 5% for G-1 in the range [0.2; 20], which covers a wide range of laboratory and industrial flows. Finally, the particular range of G and the appropriate values of c and d depend on each field and range of application of the theory.

It is worth noting that (9) (coupled with the definition of y_0 in (3)) in fact provides an approximate, developed expression for the momentum balance over the flow cross-section

$$\frac{\partial p^*}{\partial x} = \alpha \frac{\tau_p}{b} \tag{10}$$

in which τ_p is the magnitude of the shear stress at the wall ($y = b$)

$$\tau_p = \tau_c + K|\dot{\gamma}_p|^n, \tag{11}$$

where $\dot{\gamma}_p$ is the value of the shear rate at the wall. Thus expression (9) is particularly useful since it makes it possible to obtain an approximate equivalent explicit form of Darcy's law for yield-stress fluids for unidirectional flow:

$$\frac{\partial p^*}{\partial x} = \frac{\alpha \tau_c}{b}\left[1 + c\left(\frac{K|U|^n}{\tau_c b^n}\right)^d\right]. \tag{12}$$

For flows of yield-stress fluids through more complex uniform geometries it is clear from (10) and (11) that the generalized expression of (12) will be

$$\frac{\partial p^*}{\partial x} = \frac{\alpha \tau_p}{R_H} = \frac{\alpha \tau_c}{R_H}\left[1 + f(H_b)\right] \quad \text{where} \quad f(x) \to 0 \quad \text{when} \quad x \to \infty. \tag{13}$$

Here f is a positive function which depends on the geometry, and, in the expression of H_b, b must be replaced by R_H, the hydraulic radius (surface/perimeter of the channel cross-section).

Equations (12) and (13) again express the fact that, for a yield-stress fluid, the flow can take place between the plates only if the force, i.e. the pressure drop in this case, is sufficiently large. This is the basic difference between (12) or (13) and the corresponding form for Newtonian fluids

$$\frac{\partial p^*}{\partial x} = -\frac{\mu}{k_0}U, \tag{14}$$

where μ is the fluid viscosity and k_0 the 'permeability', which may be computed as a function of channel geometry. For the flow between two parallel plates k_0 is equal to $b^2/3$. In that case the corresponding expression (14) may obviously be obtained from the present analysis with yield-stress fluids, simply taking $n = 1$, $\tau_c = 0$, $K = \mu$, $d = 1$ and $c = 3$ in (12). Equation (14) is also valid for the flow of a Newtonian fluid through a porous medium and is it highly probable that an expression of the type of (13) would apply for flows of yield-stress fluids through a porous medium.

4. Flow stability of a yield-stress fluid displacing another between two parallel plates

We shall now consider the case of a yield-stress fluid pushing another yield-stress fluid of different density and viscosity between two parallel solid planes. Subscripts 1 and 2 will be used to refer to fluid 1 (at the left on the x-axis) and fluid 2 respectively. The origin of the frame of reference ($x = 0$) is taken at the average position (over y and z) of the interface between the fluids. We assume that fluid 1 pushes fluid 2. Thus U is positive ($\alpha = -1$). Owing to mass conservation U must be equal in each part. Under these conditions equations (2), (4), (5), (10) and (12) remain valid when the appropriate subscripts are added and, in particular, we have

$$p_{0j}^* = p_{0j} + \rho_j g Z = -\frac{\tau_{pj}}{b}x + B_j, \quad j = 1, 2. \tag{15}$$

For a stable flow the interface between the two fluids is a cylinder, convex in the part $y < 0$, and with a generatrix directed along z. The condition at the interface in terms of the mean pressure is simply obtained by integrating the local condition over the fluid thickness b:

$$B_1 = B_2 + \frac{\sigma_{12}}{\mathscr{R}}, \tag{16}$$

where σ_{12} is the interfacial tension between the two fluids, and $\mathscr{R}$ is an average radius of front curvature defined as

$$\frac{1}{\mathscr{R}} = \frac{1}{\mathscr{L}}\int_0^{\mathscr{L}} \frac{1}{|\mathscr{R}(y)|}\,\mathrm{d}\mathscr{L}, \tag{17}$$

where $\mathscr{R}(y)$ is the local radius of front curvature in the plane (x, y) and $\mathscr{L}$ the free surface in this plane. The set of equations (15) and (16) along with (12) makes it possible to solve the problem, i.e. find the relation between the mean velocity and the pressure drop between the upstream (fluid 1) and the downstream (fluid 2) extremities of the flow.

Let us now consider that the above flow is slightly disturbed. We assume that the form of the front is affected in the plane (x, z) but that its curvature in the plane (x, y) remains the same. As a consequence, perturbation of the advance of the front does not depend on y and takes the form

$$\eta = \varepsilon \exp(\mathrm{i}kz + \omega t), \tag{18}$$

where ε is the amplitude of the perturbation (ε is much smaller than D and L), k the wavenumber and ω the growth constant of the perturbation. The conventional treatment of Saffman–Taylor instability for Newtonian fluids is based on the perturbation of Darcy's law and mass conservation in vectorial forms (Saffman & Taylor 1958). Since we are dealing with yield-stress fluids whose constitutive equation is more complex Darcy's law has not been established in vectorial form and it seems more appropriate to start by analysing the stability in a general manner. In particular, the components of the stress tensor will be deduced from the perturbed velocity distribution and the momentum equations will be solved. Subsequently, the boundary conditions will be taken into account after integration over the fluid depth. This approach is similar to the one used by Wilson (1990) and Sader *et al.* (1994) for power-law fluids. We shall see that, with yield-stress fluids, this leads to some inconsistencies which are overcome if one only considers the results in terms of mean variables.

Within the framework of the lubrication approximation we still neglect the velocity component along y and assume that the velocity profile along directions x and z is slightly perturbed. Under these conditions we look for the local, instantaneous velocity of the form

$$u_j = u_{0j}(y) + \varepsilon\phi_j(x)\, v_j(y) \exp(\mathrm{i}kz + \omega t) \tag{19}$$

and the generalized pressure of the form

$$p_j^* = p_{0j}^*(x,t) + \varepsilon f_j(x) \exp(\mathrm{i}kz + \omega t), \tag{20}$$

where ϕ_j, v_j and f_j are *a priori* unknown functions.

On account of mass conservation the velocity component in direction z can be written

$$w_j = -v_j(y)\frac{\varepsilon\phi_j'(x)}{\mathrm{i}k} \exp(\mathrm{i}kz + \omega t). \tag{21}$$

From (19) and (21) we deduce the shear rate at leading order in the upper part of the cell ($y > 0$):

$$\left[2\mathrm{tr}(\boldsymbol{D}_j^2)\right]^{1/2} = -u_{0j}'(y) - \varepsilon\phi_j(x)\, v_j'(y) \exp(\mathrm{i}kz + \omega t) \tag{22}$$

and, within the framework of the lubrication approximation, the significant components of the stress tensor in the sheared regions for $y > 0$ are:

$$\tau_{xy} = -\left[\tau_{cj} + K_j|\gamma_j|^{n_j} - n_j\, \varepsilon K_j|\dot{\gamma}_j|^{n_j-1}\phi_j(x)\, v_j'(y) \exp(\mathrm{i}kz + \omega t)\right], \tag{23}$$

$$\tau_{yz} = \frac{\left(\tau_{cj} + K_j|\dot{\gamma}_j|^{n_j}\right)}{|\dot{\gamma}_j|}\left[-\frac{\varepsilon\phi_j'(x)}{ik} v_j'(y) \exp(\mathrm{i}kz + \omega t)\right], \tag{24}$$

where $|\dot{\gamma}_j| = -u_{0j}'(y)$. Taking into acount the definitions of u_{0j} and p_{0j}^*, the equations of motion at leading order now are

$$-f_j'(x) + n_j\phi_j(x)\frac{\partial}{\partial y}\left(K_j|\dot{\gamma}_j|^{n_j-1} v_j'(y)\right) = 0, \tag{25}$$

$$-\mathrm{i}kf_j(x) - \frac{\phi_j'(x)}{\mathrm{i}k}\frac{\partial}{\partial y}\left(\left[\frac{\tau_{cj}}{|\dot{\gamma}_j|} + K_j|\dot{\gamma}_j|^{n_j-1}\right] v_j'(y)\right) = 0. \tag{26}$$

From (25)–(26) it may be shown that $v_j(y)$ is equal to $u_{0j}(y)$. Since the perturbed velocity profile still includes an unyielded region, it may seem surprising that the front be distorted as assumed in (18). In fact this is a general problem for gradually varying flows of yield-stress fluids (Lipscomb & Denn 1984; Piau 1996). In the case of squeezing flow Wilson (1993) has shown that under particular conditions the usual theory can be retrieved as a limiting case of the bi-viscous model. We emphasize that, in the present work, the form assumed for the disturbed velocity field, though making it possible to solve the motion equations under the lubrication assumption, must not be seen as the exact, local solution of the problem. Indeed the thickness of the plug, which, as mentioned above, was already in itself a simplification even for the stable flow, should also be affected by the disturbance. Here, the disturbed velocity field is only a tool (consistent with mass conservation and motion equations, but not completely with the constitutive equation (1*b*)) to estimate the stress field from which one deduces the pressure drop (correspondingly the wall shear stress). As for the lubrication approach for a yield-stress fluid, this leads to some inconsistencies if one looks at the exact velocity field and in particular the unsheared regions. But, since this is based on the basic conservation equations and since it only involves a slight perturbation of the field, this should provide expressions for the mean variables (over the fluid thickness) consistent with experiments at leading order (as for squeeze flows).

After elimination of f_j from (25)–(26) we obtain

$$\phi_j'' - n_j k^2 \phi_j = 0 \tag{27}$$

from which it results that

$$\phi_j(x) = M_j \exp(-n_j^{1/2} kx) + N_j \exp(n_j^{1/2} kx). \tag{28}$$

M_1 and N_2 must be equal to zero since a solution for which the perturbation grows in space from its origin is not realistic. The equality of the mean velocity along the interface as given by (19) and obtained by deriving (18) with respect to the time gives

$$U\phi_j(0) = \omega \tag{29}$$

so that we finally have

$$\phi_j(x) = \frac{\omega}{U} \exp((-1)^{j+1} n_j^{1/2} kx). \tag{30}$$

Taking into account (30) and integrating (26) between y_0 and b we deduce

$$f_j(x) = (-1)^j \frac{1}{k} \frac{\omega}{U} n_j^{1/2} \frac{\tau_{pj}}{b} \exp((-1)^{j+1} n_j^{1/2} kx). \tag{31}$$

It is worth noting that this solution of the equation of motion within the framework of the lubrication approximation is also the exact solution in the form (19)–(20) of the general motion equations without inertia far from boundaries. Indeed the sum of the additional stress terms, not taken into account in the momentum balance within the framework of the lubrication approximation, is equal to zero along x and z.

The balance of pressure integrated over the thickness at the front can be written

$$p_{01}(\eta) + f_1(0)\eta = p_{02}(\eta) + f_2(0)\eta + \sigma_{12}\left(\frac{1}{\mathscr{R}} - \eta''(z)\right), \tag{32}$$

which, taking into account (15)–(16), (18) and (31), transforms as

$$\left[n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2}\right] \frac{\omega}{bkU} = -\sigma_{12} k^2 + \frac{\tau_{p2} - \tau_{p1}}{b} + g(\rho_2 - \rho_1)\left(\frac{\mathrm{d}Z}{\mathrm{d}x}\right). \tag{33}$$

Dimensionless numbers formally similar to those usually considered can be introduced: a capillary number, C, a ratio of viscous effects, B, and a gravity number, W, where

$$C = \frac{b\left(n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2}\right)}{\sigma_{12}}, \quad B = \frac{\tau_{p2} - \tau_{p1}}{n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2}}, \quad W = \frac{gb(\rho_2 - \rho_1)\mathrm{d}Z/\mathrm{d}x}{n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2}},$$

and the dispersion equation in dimensionless form is

$$\overline{\omega} = -C\overline{k}^3 + (B + W)\overline{k}, \tag{34}$$

where

$$\overline{\omega} = \frac{\omega\sigma_{12}}{U\left(n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2}\right)}, \quad \overline{k} = \frac{k\sigma_{12}}{n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2}}.$$

As a consequence the flow is unstable when

$$B + W > 0, \tag{35}$$

with a wavenumber of maximum growth

$$\overline{k}_m = \left(\frac{B + W}{3C}\right)^{1/2}. \tag{36}$$

As for Newtonian fluids, the instability mainly depends on the relative value of viscous terms through B and may be damped or enhanced by gravity effects. It can also be damped by surface tension effects. When gravity effects are negligible instability occurs when the fluid with the smallest wall shear stress (with identical mean velocity), i.e. the less viscous fluid, pushes the other. In the limit of large U, the terms containing yield stress become negligible in (33) which, for $n_j = 1$, gives the dispersion equation of the Saffman–Taylor instability for Newtonian fluids. For other values of n_j, (33) gives the equation of dispersion for power-law fluids.

More interesting within the framework of our study is the limit of small velocities for which the yielding behaviour of the fluids is predominant. For vanishing velocities, unlike Newtonian fluids (for which the unstable wavelengths can become much larger than D), instability may ocur even in the absence of gravity effects, as soon as the yield stress of the pushed fluid is sufficiently large. Indeed, for finite surface tension, B tends to a finite value when U tends to zero and the value of the wavelength of maximum growth

$$\lambda_m = 2\pi\left(\frac{3\sigma_{12}b}{\tau_{c2} - \tau_{c1}}\right)^{1/2} \tag{37}$$

is finite and depends on the value of $\sigma_{12}/(\tau_{c2} - \tau_{c1})$.

In the case of small front velocity it is also interesting to study the behaviour of the pushed fluid in the regions with the lowest velocities. Indeed, for vanishing velocities, the stress should slightly overcome the yield stress in the regions with highest velocities and as a consequence, intuitively, the stress might be smaller than the yield stress in regions with lowest velocities. In order to solve this problem we could use the expression for the stress, (23), which predicts flow everywhere (as long as $\varepsilon << 1$) since, obviously, it is based on the assumed velocity distribution obtained by slightly disturbing the stable profile. However, as discussed above, this velocity distribution is only used to estimate, via the motion equations and mass conservation, the perturbation induced on the relationship between the pressure and mean velocity. Within this framework, despite the approximations, this relationship should be relevant at leading order. On the other hand the detailed characteristics of the velocity distribution and the resulting stress components cannot constitute a solid basis for further developments.

Since it has been established by taking into account boundary conditions it is more relevant to consider the pressure drop by unit length from (15), (20) and (31) for unstable flow:

$$\frac{\partial p_j^*}{\partial x} = -\frac{\tau_{pj}}{b}\left[1 + \eta n_j \frac{\omega}{U} \exp((-1)^{j+1} n_j^{1/2} kx)\right], \tag{38}$$

where τ_{pj} is still given in the form (12). Now we assume that equation (38) is valid for a finite though small amplitude of the perturbation. Thus, since $\omega > 0$, for a fixed (sufficiently small) velocity, there can be a time after which

$$\eta > \frac{c_j}{n_j}\left(\frac{K_j\,|U|^{n_j}}{\tau_{cj}\, b^{n_j}}\right)^{d_j} \frac{U}{\omega}. \tag{39}$$

As a consequence, close to the front, after this time, $\partial p_j^*/\partial x$ is smaller than $-\tau_{cj}/b$ for the penetrating part of the fingers whereas it becomes slightly larger for the other part of the fingers. The term $-\tau_{cj}/b$ is precisely the critical value for a stable flow to take place between the planes. As a consequence the regions left behind should remain static just after the beginning of the unstable process. As long as the fingers grow, the pressure drop applied to these regions therefore decreases so that they should remain static even after a long time. From a strict point of view the linear stability analysis is only valid for $\epsilon \to 0$, which means that the last demonstration cannot be considered as perfectly consistent. Nevertheless, since it also relies on the basic characteristics of yield-stress fluids, this approach suggests a typical trend of the Saffman–Taylor instability for yield-stress fluids.

5. Radial flow of a yield-stress fluid between two parallel plates

Here we shall consider the radial flow of a yield-stress fluid from a fixed source (at O) between two parallel plates. We shall use a cylindrical frame of reference (r, θ, z) where Oz is the axis perpendicular to the plates. We assume negligible inertia and gravity effects, which in particular means that the flow part under consideration is not too close to the central point and Oz is close to the vertical axis. Within the framework of the lubrication approximation (the distance between plates ($2b$) is much smaller than the radial length of the fluid, R), the motions equations can be written

$$0 = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{\mathrm{rz}}}{\partial z}, \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}, \quad 0 = -\frac{\partial p}{\partial z}. \tag{40}$$

The velocity component along Oz, u_z, is negligible compared to the two other components u_r and u_θ, and the conservation of mass gives

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0. \tag{41}$$

Here we consider the case of a stable flow: in other words the distance between the front and the source, $R(t)$, does not depend on θ. As a consequence the velocity components do not depend on θ and u_θ is equal to zero. Under these conditions the conservation of mass gives

$$u_r = \frac{u(z,t)}{r}. \tag{42}$$

From (1*b*), (40) and (42) we have the result that p is a function of r and t only and

$$p = p_0(r,t) = \begin{cases} \dfrac{Ar^{1-n}}{1-n} + B & \text{for} \quad n \neq 1 \\ A \ln r + B & \text{for} \quad n = 1, \end{cases} \tag{43}$$

where A and B are two functions which depend on the time only and are determined from boundary conditions. By integrating (40) and upon consideration of (1*a*) it may be shown that

$$\frac{\partial u_r}{\partial z} = 0 \quad \text{for} \quad |z| \leqslant z_0 \quad \text{with} \quad z_0 = \frac{\tau_c}{\alpha \partial p/\partial r}. \tag{44}$$

The solution of the motion equation is a velocity distribution of u_r along Oz similar to that given by (4) along Oy, in which $\partial p^*/\partial x$ must be replaced by $\partial p/\partial r$ and y_0 by z_0. Under these conditions equations (5)–(12) remain valid but U now depends on r since $\partial p/\partial r = A/r^n$.

Now we consider the case of a yield-stress fluid pushing another radially at a constant flow rate. The subscripts 1 and 2 will be used to refer to the inner and outer fluids respectively. We have

$$A_j(t) = \alpha \frac{\tau_{pj}(r)}{b} r^n \approx \alpha \frac{\tau_{cj}}{b} \left[1 + c_j \left(\frac{K_j |U(R)|^{n_j}}{\tau_{cj} b^{n_j}} \right)^{d_j} \right] R(t)^{n_j}, \tag{45}$$

where α is equal to 1 for a flow towards the source and equal to -1 otherwise, and the condition at the interface is written

$$p_1(R,t) = p_2(R,t) + \sigma \left(-\frac{\alpha}{\mathscr{R}} + \frac{1}{R} \right). \tag{46}$$

The set of equations (43), (45)–(46) can be solved to deduce the mean velocity as a function of the pressure drop.

Let us consider a slight perturbation of this interface as in §4. Now the interface is situated at $R(t) + \eta$ where

$$\eta = \varepsilon \exp(\mathrm{i}k\theta + \omega t), \tag{47}$$

in which, because of mass conservation, k is an integer equal to or larger than 1. We look for the local instantaneous velocity of the form

$$u_{rj} = u_{r0j}(z,r,t) + \varepsilon \phi_j(r) \frac{v_j(z)}{r} \exp(\mathrm{i}k\theta + \omega t). \tag{48}$$

and the disturbed pressure of the form

$$p_j = p_{0j}(r,t) + \varepsilon f_j(r) \exp(\mathrm{i}k\theta + \omega t), \tag{49}$$

where ϕ_j, v_j and f_j are *a priori* unknown functions. Using the mass conservation and equation of motion in a similar manner to §4 we find that v_j is equal to ru_{r0j} and that

$$r^2 \phi_j'' + (2-n) r \phi_j' - nk^2 \phi_j = 0. \tag{50}$$

Now the kinematic condition at the interface is written at leading order

$$\phi(R) = \frac{\omega}{U} + \frac{1}{R} \tag{51}$$

so that, we finally have

$$\phi_j(r) = \left(\frac{\omega}{U} + \frac{1}{R} \right) \left(\frac{r}{R} \right)^{aj}, \tag{52}$$

$$f_j(r) = -\alpha a_j \tau_{pj} \left(\frac{\omega}{U} + \frac{1}{R}\right) \frac{R}{bk^2} \left(\frac{r}{R}\right)^{a_j+1} \tag{53}$$

where $a_j = \frac{1}{2}[n_j - 1 + (-1)^{j+1}((1-n_j)^2 + 4n_j k^2)^{1/2}]$.

The condition at the interface concerning the mean pressure is now written of (cf. Wilson 1975):

$$p_{01}(R+\eta) + f_1(R)\eta = p_{02}(R+\eta) + f_2(R)\eta - \sigma_{12}\left(\frac{\alpha}{\mathscr{R}} - \frac{1}{R} + \frac{\eta + \eta''(\theta)}{R^2}\right) \tag{54}$$

from which we deduce the equation of dispersion:

$$\left[-a_1(k)\tau_{p1} + a_2(k)\tau_{p2}\right] \frac{R}{bk^2}\left(\frac{\omega}{U} + \frac{1}{R}\right) = \alpha\sigma_{12}\left(\frac{1-k^2}{R^2}\right) + \frac{\tau^*_{p1} - \tau^*_{p2}}{b}, \tag{55}$$

where

$$\tau^*_{pj} = \tau_c \left[1 + c_j \left(1 - n_j d_j \frac{R}{U}\frac{\partial U}{\partial R}\right)\left(\frac{K_j|U|^{n_j}}{\tau_{cj} b^{n_j}}\right)^{d_j}\right].$$

For a constant flow rate (Q), $U = Q/4\pi Rb$ so that $(R/U)\partial U/\partial R = -1$.

In the limit of large values of $k(\gg 1)$ or for $n_j = 1$, i.e. for Bingham fluids, we have $a_j = (-1)^{j+1} k n_j^{1/2}$. We shall examine analytically the stability problem in that typical case, i.e. for such values of a_j. In dimensionless form (54) is

$$\overline{\omega} = \alpha C(\overline{k}^3 - \overline{k}) + B\overline{k} - 1, \tag{56}$$

where

$$\overline{\omega} = \frac{\omega R}{U}, \quad \overline{k} = kR, \quad C = \frac{\sigma_{12}\, b}{(n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2})R^2}, \quad B = \frac{\tau^*_{p2} - \tau^*_{p1}}{n_1^{1/2}\tau_{p1} + n_2^{1/2}\tau_{p2}}.$$

For $U < 0$ the flow is stable for $B > 1$, unstable for $-2C < B < 1$ with a wavelength of maximum growth equal to $2\pi R$, i.e. the centre of gravity of the circular layer of fluid should simply be displaced from its original position, and unstable for $B < -2C$ with a wavelength of maximum growth

$$\lambda_m = 2\pi R\left(\frac{3C}{C-B}\right)^{1/2}. \tag{57}$$

For $U > 0$ the flow is stable for $B < \min[3(C/4)^{1/3} - C; 1]$, and unstable otherwise, with a wavelength of maximum growth equal to $2\pi R$ when $B < 2C$, and equal to

$$\lambda_m = 2\pi R\left(\frac{3C}{C+B}\right)^{1/2} \tag{58}$$

when $B > 2C$. Like the longitudinal Hele-Shaw flow the same overall stability features as for the Newtonian case result with the same two main differences: possible instability for vanishing velocity and permanent tracks left behind.

6. Conclusion

For radial and longitudinal flows of yield-stress fluids through a Hele-Shaw cell or porous medium, the same characteristics of instability as for Newtonian fluids are found except that the wavelength of maximum growth can be small even at vanishing velocities and the flow might tend to leave a lasting trace behind the advancing front.

It is worth noting that these results rely on the assumption that the fluid does not slip at the wall whereas wall slip is often observed with yielding suspensions. Obviously, as already emphasized by Sader *et al.* (1994), stability analysis cannot predict flow characteristics a significant time after destabilization. However, the present theory have highlighted two fundamental characteristics of instability with yield-stress fluids, which, since they are intimately related to the basic mechanical properties of such fluids, should constitute basic, qualitative characteristics of the flow long after the beginning of instability.

Even within this framework it seems unlikely that these features can give rise to all the patterns mentioned for colloidal systems (Van Damme *et al.* 1994). We suggest that taking into account thixotropy along with the yielding behaviour of these systems could make it possible to explain some of these patterns. Indeed, in that case, the fluid yield stress in some regions behind the front may remain provisionally smaller than in as yet unperturbed regions. As a consequence fingering is possible within the main fingers until sufficient restructuring of the fluid has occurred. This might result in a limited branching process, with a length scale depending on a balance between the characteristic time of front advancement (which depends on the radial distance) and the characteristic time of fluid restructuring (which also depends on the previous flow). An appropriate theory for such a phenomenon would involve using existing, speculative models of constitutive equations for thixotropic suspensions.

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